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Thermal Squeezed State Representation of Scalar Field and Energy Density in Semiclassical Theory of Gravity

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The semiclassical theory of gravity is studied in terms of representation of scalar field in thermal coherent state and thermal squeezed state formalisms. For the FRW cosmological model with a minimal scalar field, the semiclassical Einstein equation reduces to zero-point energy term plus a finite temperature term and classical term in thermal coherent state. In thermal squeezed vacuum state it reduces to quantum term in addition to the finite temperature term and zero-point energy term. The present study can account for nonclassical state and finite temperature effect contributions to energy density in semiclassical theory of gravity.

1. INTRODUCTION

The semiclassical theory of gravity has been developed as a methodology to include some part of quantum effects into classical gravity. In this approach the metric under consideration is treated classically and matter field quantum mechanically as the source of gravity. In semiclassical theory left-hand side of Einstein equation contain Einstein tensor ($G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$) and right-hand side is the expectation value of suitably defined energy–momentum tensor ($\langle T_{\mu\nu}\rangle$) for matter field (Birrel and Davies, 1982). The matter field has been studied extensively over the years in terms of real as well as complex scalar field. The interplay between the field theory including finite temperature field theory and classical general relativity has been widely discussed in different contexts in cosmology. Various representation schemes of scalar field have been introduced including coherent states and squeezed states of quantum optics. The coherent state and squeezed state can play a key role for several cosmological issues. Recently such representation of scalar field received much attention in cosmology (Albrecht *et al.*, 1994; Berger, 1981; Brandenberger *et al.*, 1992, 1993; Gasperini and Giovannini, 1993; Grishcuk and

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Sidorov, 1990; Hu *et al.*, 1994; Kuo and Ford, 1993; Matacz *et al.*, 1993; Suresh *et al.*, 1995; Suresh and Kuriakose, 1997, 1998). The finite temperature effects in scalar field can play an important role in various contexts of cosmological problems at the same time the coherent states and squeezed states formalisms can also play a vital role for many problems in cosmology. Therefore it is appropriate to study the role of thermal coherent state and thermal squeezed state formalisms in cosmology and may be much useful to deal with quantum effects and thermal effects simultaneously in semiclassical theory of gravity.

In this paper we study the representation of free scalar field in thermal coherent and thermal squeezed state formalisms. Then energy density for the minimally coupled scalar field in semiclassical theory of gravity can be computed for a FRW cosmological model.

2. THERMAL SQUEEZING AND SEMICLASSICAL THEORY

Consider a classical scalar field $\Phi(x, t)$ described by the action (we take $G = \hbar = c = 1$)

$$A = \int \sqrt{-g}\, d^4x \left[\frac{1}{2}\nabla^i \Phi \nabla_i \Phi - V(\Phi)\right] \tag{1}$$

in Friedmann–Robertson–Walker spacetime with the metric

$$ds^2 = dt^2 - S^2(t)\left[\frac{dr}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\varphi)\right]. \tag{2}$$

Here k takes 1, 0, and -1 for a closed, flat, and open universe, respectively.

The classical Einstein equation is given by

$$\left(\frac{\dot{S}}{S}\right)^2 + \frac{k}{S} = \frac{8\pi}{3}\left(\frac{\dot{\Phi}^2}{2} + V(\Phi)\right) \tag{3}$$

and the classical field equation is

$$\frac{1}{S^3}\frac{d}{dt}\left[\frac{S^3\, d\Phi}{dt}\right] + \frac{dV(\Phi)}{d\Phi} = 0. \tag{4}$$

In semiclassical quantum gravity Einstein equation is (Kim, 1997)

$$\left(\frac{\dot{S}}{S}\right)^2 + \frac{k}{S} = \frac{8\pi}{3S^3}\langle\hat{H}\rangle \tag{5}$$

and the time dependent Schrodinger equation for the matter field is given by

$$i\frac{\partial}{\partial t}\Psi(\Phi, t) = \hat{H}\Psi(\Phi, t). \tag{6}$$

Where

$$\hat{H} = \frac{1}{2S^3}\hat{\Pi}^2 + S^3 V(\hat{\Phi}). \tag{7}$$

is the Hamiltoinan of the matter field.

Let us take V as following form

$$V(\hat{\Phi}) = \frac{1}{2}m^2\Phi^2. \tag{8}$$

Now the scalar field can be represent in thermal squeezed states and thermal coherent states and the semiclassical equation can be study in these formalisms. The scalar field under consideration is minimally coupled to the gravity. The present study confine attention to homogeneous modes of scalar field only. Again for the sake of simplicity of study single mode of the scalar field only consider here. Now for representing the scalar field in thermal squeezed state and thermal coherent state formalisms construction of Fock space is required which can be achived by introducing annihilation and creation operators in the following way (Kim, 1997).

$$\begin{aligned} \hat{a}^\dagger &= -i[\eta(t)\hat{\Pi} - S^3\dot{\eta}(t)\hat{\Phi}] \\ \hat{a} &= i[\eta^*(t)\hat{\Pi} - S^3\dot{\eta}^*(t)\hat{\Phi}]. \end{aligned} \tag{9}$$

Where $\hat{a}$ and $\hat{a}^\dagger$ are obeying the following relations.

$$\begin{aligned} i\,\frac{d\hat{a}^\dagger}{dt} + [\hat{a}^\dagger, \hat{H}] = 0 \\ i\,\frac{d\hat{a}}{dt} + [\hat{a}, \hat{H}] = 0. \end{aligned} \tag{10}$$

Now form the usual commutation relation it follows that

$$S^3(\eta\dot{\eta}^* - \eta^*\dot{\eta}) = i. \tag{11}$$

The position and momentum operators are given by

$$\begin{aligned} \hat{\Phi} &= (\eta\hat{a} + \eta^*\hat{a}^\dagger) \\ \hat{\Pi} &= S^3(\dot{\eta}\hat{a} + \dot{\eta}^*\hat{a}^\dagger). \end{aligned} \tag{12}$$

The density matrix approach usually gives us a convenient method for incorporating finite temperature effects. The formalism of thermo field dynamics (Umezawa *et al.*, 1982) can be used to get the thermal counterparts of coherent state (cs) and squeezed state (ss) (Shumaker, 1986). Based on these lines, thermal coherent states and thermal squeezed states are defined (Kireev *et al.*, 1989; Mann and Revzen, 1989). Here we consider single mode case only.

A thermal coherent state (tcs) is defined (Mann and Revzen, 1989) as

$$|\text{tcs}\rangle = D(\alpha)D(\overline{\alpha})\,|0(\beta)\rangle \tag{13}$$

where

$$D(\alpha) = e^{\alpha a^\dagger - \alpha^* a}$$
$$D(\overline{\alpha}) = e^{\overline{\alpha}\overline{a}^\dagger - \overline{\alpha}^*\overline{a}} \tag{14}$$

and

$$|0(\beta)\rangle = e^{-iM}\,|0, \overline{0}\rangle, \quad M = -i\theta(\beta)(a^\dagger\overline{a}^\dagger - a\overline{a}). \tag{15}$$

The density operator can be defined (Mann and Revzen, 1989) as

$$\rho(\beta;\alpha)_{\text{tcs}} = D^\dagger(\alpha)\, e^{-\beta\omega a^\dagger a} D(\alpha) \tag{16}$$

where $\beta = 1/kT$ and ω the energy of the mode, T the temperature, and k Boltzmann's constant. The characteristic function (F_{c}) for a state is defined by

$$F_{\text{c}}(\lambda\lambda^*) \equiv \langle \exp(\lambda a^\dagger \exp(-\lambda^* a)\rangle, \tag{17}$$

where

$$\langle A\rangle \equiv tr\rho A. \tag{18}$$

λ and λ^* are regarded as independent variables so that we have two parameters for each mode. Therefore the characteristic function for single mode thermal coherent state F_{ctcs} is defined by

$$F_{\text{ctcs}}(\lambda\lambda^*) = \exp[-f(\beta)|\lambda|^2 + \lambda^*\alpha - \lambda\alpha^*] \tag{19}$$

with

$$f(\beta) = \frac{1}{e^{\beta\omega} - 1}. \tag{20}$$

Similarly a thermal squeezed state (tss) is defined (Kireev *et al.*, 1989) as

$$|\text{tss}\rangle = S(\xi)S(\overline{\xi})D(\alpha)D(\overline{\alpha})\,|0(\beta)\rangle \tag{21}$$

where

$$S(\xi) = \exp\left[\frac{\xi a^{\dagger^2} - \xi^* a^2}{2}\right], \quad \xi = r\, e^{i\phi}$$
$$S(\overline{\xi}) = \exp\left[\frac{\overline{\xi}\overline{a}^{\dagger^2} - \overline{\xi}^*\overline{a}^2}{2}\right], \quad \overline{\xi} = \overline{r}\, e^{i\overline{\phi}} \tag{22}$$

and $D(\alpha)$, $D(\overline{\alpha})$, and $|0(\beta)\rangle$ are given by (14) and (15), respectively.

The density matrix for thermal squeezed states is given by (Kireev *et al.*, 1989)

$$\rho_{\text{tss}} = D^\dagger(\alpha)S^\dagger(\xi)\, e^{-\beta a^\dagger a} S(\xi)D(\alpha) \tag{23}$$

and the characteristic function is

$$F_{\text{ctss}} = \exp\left[-|\lambda|^2 \left(\sinh r \ \coth \frac{\beta\omega}{2} + f(\beta)\right) - \frac{\cosh r \ \sinh r}{2} \ \coth \frac{\beta\omega}{2}(e^{-i\phi}\lambda^2 + e^{i\phi}\lambda^{*2}) - \lambda\alpha^* + \lambda\alpha\right]. \quad (24)$$

We can write (21) by putting $\alpha = \overline{\alpha} = 0$ in analoge with zero temperature squeezed state (Shumaker, 1986) as

$$|\text{tsv}\rangle = S(\xi)S(\overline{\xi})\,|0(\beta)\rangle \quad (25)$$

and call as thermal squeezed vacuum (tsv). The corresponding density matrix is given by

$$\rho_{\text{tsv}} = S^\dagger(\xi)\, e^{-\beta a^\dagger a} S(\xi) \quad (26)$$

and the characteristic function is

$$F_{\text{ctsv}} = \exp\left[-|\lambda|^2 \left(\sinh r \ \coth \frac{\beta\omega}{2} + f(\beta)\right) - \frac{\cosh r \ \sinh r}{2} \ \coth \frac{\beta\omega}{2}(e^{-i\phi}\lambda^2 + e^{i\phi}\lambda^{*2})\right]. \quad (27)$$

Now expectation values of a, a^2, $a^\dagger$, and $a^{\dagger 2}$ can be calculated in thermal coherent state, thermal squeezed state, and thermal squeezed vacuum state formalisms by applying their corresponding characteristic function in the following relations.

$$\begin{aligned} \langle a_i^n \rangle &= \frac{\partial^n F_c}{\partial \lambda_i^{*n}}_{|\lambda_i = \lambda_t^* = 0} \\ \langle a_i^{\dagger n} \rangle &= -\frac{\partial^n F_c}{\partial \lambda_i^n}_{|\lambda_i = \lambda_i^* = 0} \end{aligned} \quad (28)$$

Therefore the expectation values of $\hat{\Pi}^2$ and $\hat{\Phi}^2$ can be computed in thermal coherent state by using (19) and (28) obtained as

$$\begin{aligned} \langle \hat{\Pi}^2 \rangle_{\text{tcs}} &= S^6(\dot{\eta}^2\alpha^2 + \dot{\eta}^{*2}\alpha^{*2} + \dot{\eta}\dot{\eta}^*(2|\alpha|^2 + 2f(\beta) + 1)) \\ \langle \hat{\Phi}^2 \rangle_{\text{tcs}} &= \eta^2\alpha^2 + \eta^{*2}\alpha^{*2} + \eta\eta^*(2|\alpha|^2 + 2f(\beta) + 1). \end{aligned} \quad (29)$$

Similarly in thermal squeezed state by using (24) and (28)

$$\begin{aligned} \langle \hat{\Pi}^2 \rangle_{\text{tss}} = S^6 \Bigg(&\dot{\eta}^2 \left(\alpha^2 - \frac{\sinh r \ \cosh r}{2} \coth \frac{\beta\omega}{2} 2\, e^{i\phi}\right) \\ &+ \dot{\eta}^{*2} \left(\alpha^{*2} - \frac{\sinh r \ \cosh r}{2} \coth \frac{\beta\omega}{2} 2\, e^{-i\phi}\right) \\ &+ \dot{\eta}\dot{\eta}^* \left(2\sinh^2 r \ \coth \frac{\beta\omega}{2} + 2f(\beta) + 2|\alpha|^2 + 1\right)\Bigg), \end{aligned} \quad (30)$$

$$\langle\hat{\Phi}^2\rangle_{\text{tss}} = \eta^2\left(\alpha^2 - \frac{\sinh r\ \cosh r}{2}\coth\frac{\beta\omega}{2}2\,e^{i\phi}\right)$$
$$+\,\eta^{*2}\left(\alpha^{*2} - \frac{\sinh r\ \cosh r}{2}\coth\frac{\beta\omega}{2}2\,e^{-i\phi}\right)$$
$$+\,\eta\eta^*\left(2\sinh^2 r\ \coth\frac{\beta\omega}{2} + 2\,f(\beta) + 2|\alpha|^2 + 1\right).$$

Therefore the scalar field can be represented in thermal coherent state and taking this in to account the semiclassical equation (5) can be written as

$$\left(\frac{\dot{S}}{S}\right)^2 + \frac{k}{S} = \frac{8\pi}{3S^3}\left[\left(\frac{1}{2S^3}S^6\dot{\eta}^2 + \frac{S^3m^2}{2}\eta^2\right)\alpha^2 + \left(\frac{1}{2S^3}S^6\dot{\eta}^{*^2} + \frac{S^3m^2}{2}\eta^{*2}\right)\alpha^{*2}\right.$$
$$\left.+\left(\frac{1}{2S^3}S^6\dot{\eta}\dot{\eta}^* + \frac{S^3m^2}{2}\eta\eta^*\right)(2\,f(\beta) + 2|\alpha|^2 + 1)\right]. \qquad (31)$$

Since the expectation value of the position operator in thermal coherent state, i.e. $\langle\text{tcs}|\hat{\Phi}|\text{tcs}\rangle = (\eta\alpha + \eta^*\alpha^*) = \Phi_{\text{cl}}$ and momentum operator, i.e. $\langle\text{tcs}|\hat{\Pi}|\text{tcs}\rangle = (\dot{\eta}\alpha + \dot{\eta}^*\alpha^*) = \Pi_{\text{cl}}$ are yield the corresponding classical values (Kim, 1997), (31) can be written as

$$\left(\frac{\dot{S}}{S}\right)^2 + \frac{k}{S} = \frac{8\pi}{3}\left(\frac{\Pi_{\text{cl}}^2}{2S^6} + \frac{m^2\Phi_{\text{cl}}^2}{2} + H_{\text{th}} + H_0\right) \qquad (32)$$

where

$$H_{\text{th}} = f(\beta)(\dot{\eta}\dot{\eta}^* + m^2\eta\eta^*) \qquad (33)$$

and

$$H_0 = \frac{1}{2}(\dot{\eta}\dot{\eta}^* + m^2\eta\eta^*) \qquad (34)$$

are respectively the finite temperature energy density and zero-point energy density contribution terms.

Similarly the representation of the scalar field in thermal squeezed state yield the semiclassical equation (5) as

$$\left(\frac{\dot{S}}{S}\right)^2 + \frac{k}{S} = \frac{8\pi}{3S^3}\left[\left(\frac{1}{2S^3}S^6\dot{\eta}^2 + \frac{S^3m^2}{2}\eta^2\right)\left(\alpha^2 - \frac{\sinh r\ \cosh r}{2}\coth\frac{\beta\omega}{2}2\,e^{i\phi}\right)\right.$$
$$+\left(\frac{1}{2S^3}S^6\dot{\eta}^{*2} + \frac{S^3m^2}{2}\eta^{*2}\right)\left(\alpha^{*2} - \frac{\sinh r\ \cosh r}{2}\coth\frac{\beta\omega}{2}2\,e^{-i\phi}\right)$$

$$+\left(\frac{1}{2S^3}S^6\dot{\eta}\dot{\eta}^* + \frac{S^3m^2}{2}\eta\eta^*\right)\left(2\sinh^2 r\ \coth\frac{\beta\omega}{2} + 2f(\beta) + 2|\alpha|^2 + 1\right)\Bigg] \tag{35}$$

which can be written as

$$\left(\frac{\dot{S}}{S}\right)^2 + \frac{k}{S} = \frac{8\pi}{3}\left(\frac{\Pi_{\rm cl}^2}{2S^6} + \frac{m^2\Phi_{\rm cl}^2}{2} + H_{\rm th} + H_q + H_0\right) \tag{36}$$

where

$$H_q = \sinh^2 r \coth\frac{\beta\omega}{2}(\dot{\eta}\dot{\eta}^* + m^2\eta\eta^*) - \frac{\sinh r\ \cosh r}{2}\coth\frac{\beta\omega}{2}\left(e^{i\phi}(\dot{\eta}^2 + m^2\eta^2) + e^{-i\phi}(\dot{\eta}^{*^2} + m^2\eta^*)\right). \tag{37}$$

$H_{\rm th}$ and H_0 are given by (33) and (34), respectively.

Since the expectation value of $\hat{\Phi}$ and $\hat{\Pi}$ in thermal squeezed vacuum becomes zero the semiclassical equation in thermal squeezed vacuum state formalism can be written as

$$\left(\frac{\dot{S}}{S}\right)^2 + \frac{k}{S} = \frac{8\pi}{3}(H_{\rm th} + H_q + H_0) \tag{38}$$

where $H_{\rm th}$, H_0, and H_q are given by (33), (34), and (37), respectively.

3. CONCLUSIONS

We have examined the finite temperature effects in semiclassical gravity for the FRW cosmological model by representing a minimal scalar field in thermal coherent state and thermal squeezed state formalisms. The semiclassical Einstein equation is found that its energy density is the sum of a classical term, a thermal term and zero-point energy term in thermal coherent state formalism. In the case of thermal squeezed state formalism the semiclassical equation is found that its energy density can be written into a quantum term in addition to that of thermal coherent state. While in thermal squeezed vacuum state the energy density for the semiclassical equation is that of quantum term and thermal term puls zero-point energy term. When r and ϕ are equal to 0 and $T = 0$, (38) reduce to vacuum state. The vacuum state can be considered as a specific coherent state (Kim, 1997) with $\Phi_{\rm cl} = \Pi_{\rm cl} = 0$. Therefore comparing the energy density in the aformentioned vacuum state with the thermal squeezed vacuum state for the semiclassical equation leads to the following facts that the nonzero contribution can arise from quantum fluctuations around $\Phi = 0$ for thermal coherent state and additional quantum fluctuation could be due to particle creation (Suresh *et al.*, 1995) in thermal squeezed vacuum state. This is one stricking difference between the thermal cohernt state and thermal squeezed state formalism representation of scalar field in semiclassical gravity. A common feature of finite temperature effect for both states can be realised by the thermal contribution term to the energy density. Therefore we may argue that if thermal squeezed state be a possible quantum state for scalar field in an early universe where quantum phenomenon are expected to play a key role, can give rise large quantum fluctuation via created particles and may lead many physical insight into early universe. As the universe expand the temperature become reduced and therefore the correleation of temperature effect and thermal squeezed state may loose. This is because the fact that the temperature and squeezing effect are strongly correlated in thermal squeezed state. Since thermal coherent state contain both classical and thermal features and thermal squeezed states contain both features of nonclassical and thermal properties, it seems that these states may be much useful than their zero temperature counter parts.

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